THE MARKET GAME WITH PRODUCTION: COORDINATION EQUILIBRIUM AND PRICE STICKINESS

Cidgem Gizem Korpeoglu (University College London)
Stephen E. Spear (Carnegie Mellon)

May 19, 2015

Abstract

In this paper, we show that the Shapley-Shubik market game model with production and the possibility increasing returns to scale technologies naturally generates indeterminate coordination equilibria with endogenous nominal price rigidities, without any need for invoking menu costs or other artificial restraints on price adjustment. As such, we suggest that the market game model may be a better micro-foundation for new Keynesian general equilibrium analysis.

1 Introduction

Contemporary macroeconomic theory rests on the three pillars of imperfect competition, nominal price rigidity, and strategic complementarity. Of these three, nominal price rigidity (aka price stickiness) has been the most important. The stickiness of prices is a well-established empirical fact, with early observations about the phenomenon dating back to Alfred Marshall. Because the friction of price stickiness cannot occur in markets with perfect competition, modern micro-founded models (New Keynesian or NK models, for short) have been forced to abandon the standard Arrow-Debreu paradigm of perfect competition in favor of models where agents have market power and set market prices for their own goods. Strategic complementarity enters the picture as a mechanism for explaining the kinds of coordination failures that lead to sustained slumps like the Great Depression or the aftermath of the 2008 financial crisis. Early work by Cooper and John [1988] laid out the importance of these three features for macroeconomics, and follow-on work by Ball and Romer [1991] showed that failure to coordinate on price adjustments could itself generate strategic complementarity, effectively unifying two of the three pillars.

Not surprisingly, the Ball and Romer work was based on earlier work by a number of authors (see Mankiw and Romer’s New Keynesian Economics [1991]) which used the model of Dixit and Stiglitz of monopolistic competition as the
basis for price-setting behavior in a general equilibrium setting, combined with
the idea of menu costs – literally the cost of posting and communicating price
changes – and exogenously-specified adjustment time staggering to provide the friction(s) leading to nominal rigidity. While these models perform well in explaining aspects of the business cycle, they have only recently been subjected to thorough empirical testing, because of the scarcity of good data on how prices actually change. This has changed in the past decade as new sources of data on price dynamics have become available, and as computational power capable of teasing out what might be called the "fine structure" of these dynamics has emerged. On a different dimension, the overall suitability of monopolistic competition as the appropriate form of market imperfection to use as the foundation of the new macro models has been largely unquestioned, though we believe this is largely due to the tractability of the Dixit-Stiglitz model relative to other models of imperfect competition generated by large fixed costs or increasing returns to scale not due to specialization.

In this paper, we examine both of these underlying assumptions in light of what the new empirics on pricing dynamics has found, and propose a different, and we believe, better microfoundation for New Keynesian macroeconomics based on the Shapley-Shubik market game.

2 Empirical Evidence

As we noted in the Introduction, the observations on price stickiness go back to Alfred Marshall and contemporaries, but until recently, there was little detailed empirical examination of the phenomenon. So, we begin by reviewing the findings of some of this recent work, based largely on the survey by Nakamura and Steinsson [2013], and the stylized facts of nominal rigidity put forth by Klenow and Malin [2010].

Klenow and Malin, in summarizing the findings of their empirical research on price changes, come up with the following set of stylized facts about price changes:

- Prices change at least once a year on average
- Sales and product turnover are important to micro price flexibility
- There is substantial heterogeneity in the frequency of price changes across goods
- Relative price changes are transitory
- Price changes are typically not synchronized across the business cycle

Nakamura and Steinsson’s survey supports this characterization. They note, in particular, the phenomenon of "reference prices" (per Eichenbaum, Jaimovich, and Rebelo [2008]’s analysis of supermarket scanner data). These are prices which are highly persistent, don’t change frequently, with deviations
from them occurring during sales. These prices are readily apparent in the
time-series data in the EJR paper. The heterogeneity in the frequency of price
changes across goods is also apparent from a histogram of these frequencies from
Nakamura and Steinsson [2008].

We think this data is significantly at odds with what would occur if the
data were being generated by the menu costs mechanism of the Dixit-Stiglitz
based NK models. We will examine this issue in more detail below, but a key
thing to note from these two graphs is the apparent asymmetry in the time-
series data between lowering prices and raising them. If the only friction at
work in restraining price changes were the communication costs that underlie
the menu costs models, we would expect these to operate symmetrically. What
we see in the time-series data, however, is a persistence in what EJR identified
as reference prices, with slow and relatively infrequent upward adjustments,
coupled with much more frequent large downward price changes (i.e. sales).
For the distributional data, since the costs of communicating price variations
should not be significantly different for peanut butter than it is for cars, the
heterogeneity of price change frequencies that Nakamura-Steinsson and others
observe suggests that something other than just the costs of letting customers
or rivals know about price changes must be at work here. And, while models of
staggered price setting can match this kind of data, the exogenous imposition
of this behavior is hardly satisfactory.

There is a final empirical issue that arises with respect to the behavior of
the markup of price over marginal cost. The standard Dixit-Stiglitz based
NK models without menu costs of shocks predicts (see the following section for
details) that markups should be constant. The data, on the hand, shows clear
fluctuations in markups with evidence of procyclic variations, as is apparent from the chart below taken from Nekarda and Ramey’s "The Cyclic Behavior of Price-Cost Markup" [2013]. As with the response to price changes, NK models with menu costs, staggered adjustment frictions and exogenous shocks can be made to fit the markup data, but it is the frictions that drive the dynamics of the markup. We will show that these frictions aren’t necessary to obtain dynamic markup responses in the market game model.

3 New Keynesian Models

Standard New Keynesian models are based on the Dixit-Stiglitz model of monopolistic competition and typically employ a CES aggregator function in which a consumer good \( y \) is produced using intermediate goods \( y_i \) via the production function

\[
y = \left[ \int_0^1 y_i^{\frac{1}{\varepsilon}} \right]^{\varepsilon} dt.
\]

The monopolistically competitive market structure here means that firms producing output goods will take the prices of both their inputs and outputs as given, while the firms producing inputs will have some market power (via the product differentiation assumption) and hence, will act as price setters. A straightforward analysis of the output firm’s profit maximization problem yields the following relationship between the input demand, given an output level \( y \).
Source: Authors’ calculations using quarterly data from the BLS and BEA.
Notes: The BLS markup is the inverse of labor share in private business. The markups for nonfinancial corporate business are constructed by dividing NIPA data on either total compensation or wage and salary disbursements by income without capital consumption adjustment less indirect business taxes. Shaded areas represent periods of business recession as determined by the National Bureau of Economic Research.
and the prices for output $P$ and input prices $P_j$ for input good $j$

$$y_j = \left( \frac{P_j}{P} \right)^{\frac{1}{\alpha}} y.$$

If we now also impose the assumption that the production of input goods by wholesale firms exhibits constant returns to scale with no fixed costs, then the monopolistically competitive input supplier $j$ will take this demand as given and maximize her profit. Again, some well-studied optimization algebra yields the following optimal input price for firm $j$

$$P_j = \varepsilon P m_j$$

where $m_j$ is firm $j$’s (constant) marginal cost. This then yields a relative price for firm $j$’s output

$$p_j = \frac{P_j}{P} = \varepsilon m_j$$

and hence, firm $j$’s optimal markup over marginal cost is

$$\frac{p_j}{m_j} = \varepsilon$$

which is constant.

We see, then, that the simple DS model does give rise to optimal price-setting behavior in equilibrium. To convert this model into something that might explain the empirical nominal rigidity data and the possibility of coordination equilibria, however, requires adding additional frictions which are not inherent to the model, in order to prevent prices from immediately adjusting to market shocks, or to force this adjustment to be staggered over time. This is the role of the menu cost assumption and assumptions on staggered adjustment schedules in the NK literature.

The menu cost friction arises from exactly what it sounds like it arises from: it is costly for firms to communicate changes in prices. Communication costs can arise from a variety of different sources, whether it is printing new menus (the restaurant metaphor), reposting prices in outlets, advertising over various media, or reprogramming accounting systems to take account of new prices. This simple characterization of the menu cost friction suggests an equally simple (but plausible) quadratic adjustment specification of the menu cost as

$$\text{menu cost} = \frac{\gamma}{2} \left( \frac{P_t}{P_{t-1}} - 1 \right)^2.$$ 

The specification is simple enough to generate price stickiness, but much of the plausibility of the specification is due to the symmetry it exhibits with respect to price increases or decreases. Indeed, if the only reason for postponing price changes is the cost of communicating the changes to customers, there should be a symmetry between increases and decreases, and this is captured by the quadratic adjustment cost specification.
The question of cyclic fluctuations in markups is partially addressed by the imposition of price stickiness, in the sense that the relative price of output in a DS model is now fixed, but to match the data on variation in observed markups, one must make assumptions on how the marginal cost varies. In a monopolistic competition setting, this assumption must necessarily be one of increasing marginal cost (i.e. a convex cost technology) in order to be compatible with the continuum of firms assumption (which is not compatible with internalized increasing returns to scale). This assumption, in turn, leads to a prediction of counter-cycle variation in markups, which has been one of the accepted empirics of New Keynesian theory. But the issue of what is actually in the data is more complicated, since economists rarely observe marginal costs directly, and most inferences of marginal costs are based on observations of average cost, or are estimated indirectly based on modeling assumptions. More recent studies of this issue, however, have reversed the conventional empirical wisdom, concluding that, properly measured, markups are weakly procyclic (see Nakarda and Ramey [2013] and citations therein for details). This poses some obvious problems for the conventional New Keynesian models.

We will show in this paper how the Shapley-Shubik market game with increasing returns to scale firms can circumvent both the problem of asymmetric price adjustment, and procyclic markups.

4 The Market Game Model

We work with a fairly standard version of the Shapley-Shubik market game with production along the lines first considered by Dubey and Shubik [1977], but with modifications that make the model more an extension of the one consider by Peck, Shell and Spear [1992]. In this section we describe the model formally.

4.1 Agents

There are two types of agents: \( M < \infty \) standard consumers endowed with primary factors of production \( e_i \in \mathbb{R}_+^N \) only, who sell these factors to firms that produce outputs of consumer goods that enter in the preferences of consumers. For simplicity, we assume that households get no direct utility from the consumption of input goods. We impose this assumption at this point primarily as a simplification of the model and notation, though we will revisit the issue later in the analysis.

Preferences of standard consumers are as in Peck, Shell, and Spear [1992], defined over consumption goods vectors \( x_i \in \mathbb{R}_+^J \). There are \( K_j < \infty \) producers of finitely many types \( j = 1, \ldots, \ell \) who produce outputs of consumption good \( j \) using a production technology specified by a production function \( q_{kj} = f_j\left(\phi_{kj}\right) \) where \( \phi_{kj} \in \mathbb{R}_+^N \) is the vector of inputs for producer \( k = 1, \ldots, P_j \) in production sector \( j = 1, \ldots, J \). We assume that consumers are exogenously endowed with ownership shares of each firm. Specifically, we let \( \theta_{kj}^i \) be consumer \( i \)'s ownership share of firm \( k_j \) in sector \( j \).
4.1.1 Producer actions

Producers in the model purchase input commodities from the households and use these to produce output, which they then sell back to households based on their expectations of prices they can receive for their output. Since producers are not endowed with primary factors, they must bid for these on the trading posts (which are endemic to the market game) for primary factors. We assume that producers act to maximize profits\(^1\). For the time being, we let \(p_j\) be the price of output good \(j\), and \(r_n\) be the price of input good \(n\), and \(r^T = [r_1, ..., r_N]\) the vector of input prices. Profit for firm \(k_j\) is then

\[
\pi_{k_j} = p_j f_{k_j} (\varphi_{k_j}) - r \cdot \varphi_{k_j}
\]

Prices for inputs are determined on the trading posts for inputs. Producer \(k_j\)'s bid on trading post\(^2\) \(n = 1, ..., N\) is denoted \(w_{k_j}^n\), and we let \(w^T_{k_j} = \left[ w_{k_j}^1, ..., w_{k_j}^N \right] \in \mathbb{R}^N\) denoted the producer's vector of bids for inputs. The aggregate bid across all producers for input good \(i\) is

\[
W^n = \sum_{j=1}^{K} \sum_{k_j} w_{k_j}^n.
\]

As is standard, we let the aggregate bid on trading post \(i\) except for that of producer \(k_j\) be denoted \(W^n_{-k_j}\). The price of input good \(n\) is then defined as

\[
r^n = \frac{W^n}{E^n}
\]

where \(E^n = \sum_{i=1}^{M} e_i^n\). Each producer’s allocation of input goods is given by the producer’s own bid for the input divided by the price

\[
\varphi_{k_j}^n = \frac{w_{k_j}^n}{r^n} = w_{k_j}^n \cdot \frac{E^n}{W^n}.
\]

This, of course, is just the standard market game rule that allocates a share of the aggregate offer to an agent in the same proportion as that agent’s bid on the trading post is to the aggregate bid. Producers earn unit of account revenues from the sale of their outputs on the trading posts for output goods. With \(q_{k_j}^j = f_j (\varphi_{k_j})\), \(j = 1, ..., \ell\), a firm in output sector \(j\) will offer all of its

\(^1\)It is well-known that in the absence of perfect competition, shareholders can disagree on what objective the firms they own should pursue. We note below in our analysis of the households’ best-responses that this can occur here, and hence, we simply adopt the assumption of profit maximization as each firm’s objective.

\(^2\)The fictional trading posts introduced by Shapley and Shubik are essentially a metaphor for the flows of expenditure and product between traders. By collecting these flows for specific markets on trading posts, it simplifies the actions of choosing demand and supply allocations and streamlines the exposition of the market game form. In equilibrium, though, it is only the flows of expenditure and product that matter, not where they take place.
output on the trading post, so we can define the aggregate offer of good \( j \) as

\[
Q^j = \sum_{k_j=1}^{K_j} q_{kj}^j.
\]

As before, we let \( Q^j, k_j \) be the total offer of good \( j \) less that of producer \( k_j \). Given the output price \( p_j \) for good \( j \), producer \( k_j \) can spend \( p_j q_{kj}^j \) units of account on the purchase of input goods. Hence, producer \( k_j \) faces a budget constraint for bids on inputs of the form

\[
\sum_{n=1}^{N} w_n^{i_k} \leq p_j q_{kj}^j.
\]

Note that from the allocation rule, if we substitute for \( b_{n}^{i_k} \) in the expression above, we obtain the statement that profits can’t be negative. In the imperfectly competitive setting of the market game, we can expect this constraint to not bind frequently.

### 4.1.2 Consumer actions.

Consumers bid on trading posts for output goods of the firms in the second subperiod of the game. As noted above, consumers will offer their full endowment for sale on the trading posts for inputs, and will thus, given the timing structure of the game, receive income from the sale of endowments of

\[
r \cdot e_i = \sum_{n=1}^{N} r_n e_i^n = \sum_{n=1}^{N} \frac{W_n}{E_n} e_i^n
\]

where the aggregate bids and offers on the input markets are determined by the producers. In addition to their income from the sale of inputs, consumers also receive the shares of the producer firms profits they are endowed with, so that consumer \( i \)'s total income is

\[
y_i = \sum_{n=1}^{N} \frac{W_n}{E_n} E_i^n + \sum_{j=1}^{K_j} \sum_{k_j=1}^{\ell} \theta_{i}^{kj} \pi_{k_j}.
\]

Note that with a small number of consumers, given the arbitrary distribution of ownership shares across consumers, it can occur that consumers will want firms they own to deviate from profit maximization in order to increase the value of the household’s sales of primary factors. As noted above, it is also possible that households will disagree on the exact objective of each firm, so, as we did above in assuming that firms act to maximize profit, we assume here that households simply take the value of their endowments as given. This can be justified more rigorously by assuming that there are many consumers relative to firms, so that the ratio \( \frac{\theta_{i}^{kj}}{\pi_{k_j}} \) is negligible. Given these assumptions, we let consumer \( i \)'s bid on trading post \( j \) be denoted \( b_{i}^j \) and define the aggregate bid on trading post \( j \) as

\[
B^j = \sum_{i=1}^{M} b_{i}^j.
\]
As above, we let the aggregate bid on trading post \( j \) except for that of consumer \( i \) be denoted \( B^j_i \). The price of output good \( j \) is then defined as the ratio of the total bids for the good to the total offer of the good

\[ p_j = \frac{B^j}{Q^j}. \]

Consumer \( i \)'s allocation of consumption good \( j \) is then given by

\[ x^j_i = \frac{b^j_i}{p_j} = \frac{b^j_i}{B^j_i} Q^j, \]

which, again, is the usual market game allocation rule that gives each consumer the same proportion of the aggregate offer of the good as their bid is to the aggregate bid. Consumer \( i \)'s bids for output goods in terms of units of account are then constrained by the budget relation

\[ \sum_{j=1}^\ell b^j_i \leq y_i = r \cdot e_i + \sum_{j=1}^\ell \sum_{k_j=1}^{K_j} \theta_i^{k_j} \pi_{k_j} = \sum_{n=1}^N r^n e^n_i + \sum_{j=1}^\ell \sum_{k_j=1}^{K_j} \theta_i^{k_j} \pi_{k_j}, \]

\[ = \sum_{n=1}^N \frac{W^n}{E^n} e^n_i + \sum_{j=1}^\ell \sum_{k_j=1}^{K_j} \theta_i^{k_j} \pi_{k_j}. \]

### 4.1.3 Strategic interactions

On the producer sides, all producers will take the aggregate offer of inputs as given in making their optimal profit maximizing choices of output, but will take account of other producer’s (including those of other sectors) bids for inputs into account, since these affect both the firm’s cost of production, and the output price via the quantity produced. Hence, a producer’s best response to other producers’ actions is determined by the solution to the optimization problem

\[ \max_{b_{kj}} \frac{B^j}{Q^j} f_{kj} (\varphi^{kj}) - \sum_{n=1}^N \frac{W^n}{E^n} \varphi_n^{kj} \]

subject to

\[ \sum_{n=1}^N w_{nk_j}^n \leq p_j q_{kj}, \]

\[ \varphi_n^{kj} = w_{nk_j}^n \frac{E^n}{W^n} \]

\[ q_{kj} = f_{kj} (\varphi_{kj}). \]
Consumers offer all of their endowments on the trading posts for input goods, and make bids on output goods trading posts, and we assume that this offer is not a strategic variable for any individual. Hence, they will take account of the effects that their own bids for output goods have on their final allocations, and hence, their best responses are determined by the solution to the optimization problem

$$\max_{b_i} u_i [x_i]$$

subject to

$$x^j_i = b^j_i \frac{Q^j}{Q^j} \text{ for } j = 1, \ldots, \ell$$

$$\sum_{j=1}^{\ell} b^j_i \leq \sum_{n=1}^{N} \frac{W^n}{E^n} e^n_i + \sum_{j=1}^{\ell} \sum_{k_j=1}^{K_j} \theta^{k_j}_i \pi_{k_j}$$

$$\pi_{k_j} = \frac{B^j}{Q^j} f_{k_j} (\varphi^{k_j}) - r \cdot \varphi^{k_j}.$$  

4.1.4 Best Responses

On the producer side, we note first that the second constraint in the producer’s optimization is simply the requirement that profits be non-negative. We assume in the analysis below that this constraint doesn’t bind, though it is necessary in principle, for example, with increasing returns to scale technologies. Making substitutions from the other constraints into the producer’s optimization, we obtain an unconstrained optimization

$$\max_{b_{k_j}} \frac{B^j}{Q^j} f_{k_j} \left[ \left( w^1_{k_j} \frac{E^1}{W^1}, \ldots, w^N_{k_j} \frac{E^N}{W^N} \right) - \sum_{n=1}^{N} w^n_{k_j} \right].$$

Taking first-order conditions yields

$$\frac{B^j}{Q^j} \frac{\partial f_{k_j}}{\partial \varphi^{k_j}} \left[ \frac{E^n}{W^n} - \frac{w^n_{k_j} E^n}{(W^n)^2} \right] - \frac{B^j f_{k_j}}{(Q^j)^2} \frac{\partial f_{k_j}}{\partial \varphi^{k_j}} \left[ \frac{E^n}{W^n} - \frac{w^n_{k_j} E^n}{(W^n)^2} \right] - 1 = 0$$

or

$$\frac{B^j}{Q^j} \frac{\partial f_{k_j}}{\partial \varphi^{k_j}} \frac{E^n W_{-k_j}^n Q^j_{-k_j}}{(W^n)^2} - 1 = 0$$

or

$$\frac{p^n}{r^n} \frac{\partial f_{k_j}}{\partial \varphi^{k_j}} \frac{W_{-k_j}^n Q^j_{-k_j}}{Q^j} - 1 = 0.$$  

Note that if the market is such that we have a very large number of producers, the two ratios on the right-hand side of the first term will be almost one, and the expression here boils down to the statement that the value of the marginal product of the $n^{th}$ input should equal it’s price.
For the consumer’s best-response optimization, we consider

$$\max_{b_i} u_i \left[ b_i^1 Q^1, \ldots, b_i^\ell Q^\ell \right]$$

subject to

$$\sum_{j=1}^\ell b_i^j \leq \sum_{n=1}^N \frac{W_n}{E_n} c_n + \sum_{j=1}^\ell \sum_{k_j=1}^{K_j} \frac{B_j^j}{Q^j} f_{k_j} \left( \varphi^{k_j} \right) - r \cdot \varphi^{k_j}.$$

Consumers will take the bids on the input markets as given, and hence will also take the firms’ outputs as given, although they will take account of the effect their own bids on the product markets has on the prices in the profits of each firm. The first-order conditions for this optimization problem are

$$\frac{\partial u_i}{\partial x_i^j} \left[ \frac{Q^j B^j}{B^j} - \frac{b_i^j Q^j}{\left[ B^j \right]^2} \right] + \lambda \left[ \frac{\sum_{k_j=1}^{K_j} \theta_i^{k_j} f_{k_j}}{Q^j} - 1 \right] = 0$$

or

$$\frac{\partial u_i}{\partial x_i^j} \left[ \frac{Q^j B^j_{-i}}{B^j B^j} \right] + \lambda \left[ \frac{\sum_{k_j=1}^{K_j} \theta_i^{k_j} f_{k_j}}{Q^j} - 1 \right] = 0.$$

We do not need to consider the effect of a change in household $i$’s bid on input prices because of the envelope theorem as applied to firms’ profit maximizations. This follows from the fact that we can view household $i$’s income as its share of aggregate profit net of it’s own sales of primary factors. If firms are not maximizing this, then some firm is not maximizing profit, and hence the assertion follows. Finally, note that the household first-order condition is also consistent with what we get in the competitive limit, since the right-hand term will go to $-\lambda$ as the number of firms and consumers gets large, while the left-hand side goes to the marginal utility divided by the price.

As was the case in the Peck, Shell and Spear paper on pure exchange market game economies, we can show the equivalence a firm’s optimization in terms of bids and one in which it chooses inputs directly. To see this, note that

$$\varphi^{k_j} = \hat{W}^{-1} \hat{E} \hat{w}_{k_j}$$

from the allocation rule. Here the $^>$ notation indicates turning the vectors $W$, $\Omega$ of total bids and offers respectively into square $n \times n$ diagonal matrices with the vectors embedded down the main diagonal. The vector $w_{k_j}$ is firm $k_j$’s bid on the markets for input goods. Since $W$ includes this firm’s bids, we can write the allocation rule as

$$\left[ \hat{W}_{-k_j} + \hat{w}_{k_j} \right] \varphi^{k_j} = \hat{E} \hat{w}_{k_j}$$

$$\Rightarrow \hat{W}_{-k_j} \varphi^{k_j} = \left[ \hat{E} - \varphi^{k_j} \right] \hat{w}_{k_j}$$

$$\Rightarrow \hat{w}_{k_j} = \left[ \hat{E} - \varphi^{k_j} \right]^{-1} \hat{W}_{-k_j} \varphi^{k_j}.$$
Substituting this expression into the budget constraint then yields

\[ t^T w_{kj} = t^T \left( \hat{E} - \hat{\varphi}_{kj} \right)^{-1} \hat{W}_{-kj} \hat{\varphi}_{kj} \leq p^j f_{kj} (\varphi_{kj}) \]

where \( t^T = [1, ..., 1] \) is the sum vector. With this constraint, the firm’s first-order conditions for profit maximization are (after a straight-forward differentiation and algebraic manipulation)

\[ p^j \frac{Q_{-kj}}{Q^j} D f_{kj}^T - t^T \left( \hat{E} - \hat{\varphi}_{kj} \right)^{-2} \hat{W}_{-kj} \hat{E} = 0. \]

To show that this is equivalent to the FOCs obtain from the bid-based optimization, note that the allocation rule implies that

\[
\begin{align*}
\hat{\varphi}_{kj} &= \hat{W}^{-1} \hat{E} w_{kj} \\
\hat{E} - \hat{\varphi}_{kj} &= \hat{E} - \hat{W}^{-1} \hat{E} \hat{w}_{kj} \\
&= \hat{E} \hat{W}^{-1} \left[ \hat{W} - \hat{w}_{kj} \right] \\
&= \hat{E} \hat{W}^{-1} \hat{W}_{-kj}.
\end{align*}
\]

Hence,

\[
\begin{align*}
\left[ \hat{E} - \hat{\varphi}_{kj} \right]^{-2} \hat{W}_{-kj} \hat{E} &= \left[ \hat{E} \hat{W}^{-1} \hat{W}_{-kj} \right]^{-2} \hat{W}_{-kj} \hat{E} \\
&= \hat{E}^{-1} \hat{W}^2 \hat{W}_{-kj}^{-1} \\
&= \hat{p} \hat{W} \hat{W}_{-kj}^{-1}
\end{align*}
\]

and the two first-order conditions are the same.

5 Coordination Equilibria and Nominal Rigidities

In a previous paper [2015] we showed conditions for existence of Nash equilibrium for a production market game with increasing returns to scale technologies. Unlike the case for models with strictly convex technologies, there are no strong existence results for the model in the presence of increasing returns to scale technologies, for the simple reason that the non-negativity constraint on profits can become binding in the the model if there are too many IRTS firms in the market. For our purposes here, then, we will simply assume that the market configuration is such that there are IRTS firms in each production sector together with standard constant or decreasing returns to scale firms, and that the Nash equilibrium associated with the aggregate input resource endowment \( E \) exists.
5.1 Coordination Equilibria

The simplest way to demonstrate that the production model exhibits the same indeterminacy of equilibrium found in pure exchange models (see Peck-Shell-Spear [1992]) is to start from the Nash equilibrium associated with the sell-all game with aggregate endowment vector $E$. Let $E' \gg E$ be a new aggregate endowment vector which is strictly bigger than $E$ in all components, and assume that $e'_i \gg e_i$ for all households $i = 1, ..., M$. Let firms expect households to offer the same aggregate inputs $E$ when the input state is $E'$. If the firms expect the same prices for outputs and inputs to prevail in this situation, they will wish to produce the same outputs, and will hire the same inputs as in the initial equilibrium. If households off $E$ in aggregate and make the same bids for produced goods as in the initial equilibrium, we will obtain the same Nash equilibrium associated with $E'$ as that associated with $E$. The only difference here is that households could be made better off if the economy could coordinate on a new sell-all equilibrium at $E'$.

The more technical way to show this result is to construct a sequence of economies $\xi^n$, $n = 1, 2, ...$ consisting of the same households and firms, with the only difference being that for every finite $n$, households have positive marginal utilities for input goods, while in the limit, these marginal utilities go continuously to zero and the limit economy is our Hechsler-Ohlin version of the model. For finite $n$, then, households will wish to actively bid to purchase input goods for direct consumption. This setting is then completely analogous to that considered in Peck-Shell-Spear, so that households must be constrained to offer some fixed, but indeterminate, amount of input goods on the markets. (Alternatively, households could choose how much of the input goods to offer, but then their bids would have to be pre-specified at some fixed level). As in the Peck-Shell-Spear analysis, for each finite $n$ we get a Nash equilibrium for each level of offers of input goods. The equilibrium depends continuously on the input goods offers, so, in the limit, we obtain the indeterminacy of equilibrium with respect to input goods offered even though in the limit, household get to no utility value from the consumption of the inputs directly. The continuum of equilibria so attained are called the coordination equilibria, and they arise from the strategic complementarity associated with the fact that allocations are Pareto rankable across coordination equilibria.

5.2 Price Stickiness and Market-Share Equilibration

Given the coordination equilibria result, the market game exhibits a rather remarkable emergent phenomenon not present in the competitive limit, which is the fact that locally around any Nash equilibrium associated with less than full use of input resources, firms can adjust output to use more or less input resources keeping prices for all commodities fixed, by changing outputs in ways that reallocate market shares between firms on both output and input markets.
To see this, we start with the first-order conditions for a typical firm

\[ p^j \left( \frac{Q^j - k_j}{Q^j} \right) D_{\phi} f_{kj} - WW_{-kj}^{-1} r = 0 \]

written in vector form, so that \( \hat{W} \) is a diagonal matrix with aggregate bids for inputs down the main diagonal, and \( \hat{W}_{-kj} \) is a diagonal matrix with the bids of all firms but \( k_j \) for inputs on the main diagonal. Also, \( \frac{Q^j - k_j}{Q^j} \) is one minus firm \( k_j \)'s market share on the output markets, and for each input good \( n, \frac{W^n}{WW_{-kj}} \) is one minus firm \( k_j \)'s share of type \( n \) inputs. The key thing to note here is that we can vary these market shares independently across all the firms, as long as the shares all add up to one. Hence, for firm \( k_j \) being able to vary its market share looks locally like a variation in the price in the first-order condition for profit maximization. Similarly, being able to vary the input shares in the right-hand term looks like a variation in the vector of input prices. To the extent that we can make a full-rank perturbation using the market shares, then, firms can adjust to maintain Nash equilibrium without changing either output or input prices. This is the sense in which the model may give rise to sticky prices; indeed, if the market share equilibration mechanism is effective, firms can maintain prices for any reason, whether it is menu costs, reluctance to alienate suppliers or customers, or other unspecified frictions.

To show this result formally, we need to consider the constraints imposed on the perturbation by the fact that adjustment takes place via market shares, and the fact that we are holding prices constant.

The ability to mimic price effects via market shares is constrained by the fact that market shares must sum to one, and, with prices for outputs constant, product market adjustments are limited to movements along each consumer’s income expansion path; hence, for this analysis, we will assume that there are many households so that demand in the product markets is competitive. Since market shares enter the firm’s first-order conditions as one minus market share, the adding up requirement means that we have \( (J - 1)N \) free input bill share parameters, and \( \sum_{j=1}^{\ell} (K_j - 1) \) free product market share parameters. The fact that with prices fixed, adjustments on the product market are restricted to income effects means that adjustments across coordination equilibria with prices fixed can occur only for relaxations of the total input use constraint in response to some external effect on aggregate demand.

These two restrictions pose some interesting questions

- Given that we would generally need \( \ell \) output price variables and \( N \) input price variables in conjunction with consumer choices and firm input allocations to find equilibrium, how does the number of firms affect the equilibrium calculation when prices are fixed?

- Can adjustments in firm market shares in response to exogenous shocks to demand maintain the economy in Nash equilibrium? If so, does the model impose any restrictions for equilibration to work?
• Given the restriction to generating equilibration via income effects on the product market, we will clearly need to make adjustments in households’ bids, as well as firms bids on the input markets in order to keep prices fixed. What does this require in terms of adjustment in the units of account we are using to denominate prices? What implications might this have for monetary policy adjustments in a real world environment?

To address the first question, we need to write down the equilibrium conditions we wish to maintain when total resources vary. These are the first-order conditions

\[ Du_i (x_i) - \lambda_i p = 0 \]

\[ p \cdot x_i = r \cdot e_i + \sum_{j=1}^{\ell} \sum_{k_j} \theta_i^{k_j} \pi_{k_j} \]

for households; and

\[ p^j \left( \frac{Q^j}{Q^i} \right) D_{\psi_j f_{k_j}} - \hat{W} \hat{W}^{-1} r = 0 \]

\[ \theta^{k_j} = \hat{w}_{k_j} \hat{W}^{-1} E = \left( I - \hat{W}_{-k_j} \hat{W}^{-1} \right) E \]

for firms. In addition, we need to add the market-clearing and price consistency rules

\[ \sum_{i=1}^{M} x_i - \sum_{j=1}^{\ell} \sum_{k_j} \tilde{y}_{k_j} = 0 \]

where

\[ \tilde{Q}_{k_j} = \begin{bmatrix} 0 \\ \vdots \\ q_{k_j} \\ \vdots \\ 0 \end{bmatrix} \]

and

\[ p - \hat{B} \hat{Q}^{-1} t = 0 \]

\[ r - \hat{W} \hat{E}^{-1} \hat{l} = 0 \]

where \( t^T = [1, ..., 1] \) is the sum vector, and, as before, \( \hat{\cdot} \) over an aggregate variable denotes a diagonal matrix with the embedded vector down the main diagonal. In this expression, the aggregate bids on the product and input markets are taken as variables, since they can be varied independently of market shares.
This yields a system of $M (\ell + 1) + J N + \ell + N$ equilibrium conditions, in $M (\ell + 1) + (J - 1) N + \sum_{j=1}^{\ell} (K_j - 1) + \ell + N$ variables. So, to have at least as many variables as equations, then, we need

$$(J - 1) N + \sum_{j=1}^{\ell} (K_j - 1) \geq J N + \ell$$

or

$$\sum_{j=1}^{\ell} (K_j - 1) \geq N + \ell.$$

If we let $F$ be the average number of firms per output sector, the inequality above becomes

$$(F - 1) \ell \geq N + \ell$$

which yields

$$F \geq 2 + \frac{N}{\ell}$$

To get a rough estimate of this number, we use data compiled by Miroudot, Lanz and Ragoussis [2009]. From their table 5Using the sum of consumption and capital outputs to estimate $\ell$, with the intermediate goods representing $N$, we obtain $\frac{N}{\ell} = 1.6$, so that we would need $F > 4$.

The answer to the second question involves two steps.

- We need to show first that firms will in fact have incentives to respond to changes in market shares in ways that move the economy back toward equilibrium after a shock; this turns out to require some assumptions;
We then need to show what amounts to a full-rank condition for applying the implicit function theorem in order to formally show the result.

To show conditions under which firms will have incentives to respond to market share changes, consider a simple one-sector version of the model with a single output produced using a single input, and suppose that all firms in this economy produce using a strictly diminishing returns technology. Suppose we are at a slack coordination equilibrium, and the demand for the output good increases. What happens if some firm responds to the demand shock by increasing its own output, other firms holding output constant? The relevant first-order condition to answer this question is

\[ p \frac{Q}{Q} \frac{k}{k} f'(\phi) - r \frac{W}{W_k} = 0 \]

or

\[ f'(\phi) = \frac{r}{p} \frac{W}{W_k} \frac{Q}{Q_k} \frac{k}{k}. \]

If the firm increases output, its market share of output goes up, so one minus market share goes down. Similarly, its share of the wage bill will go up, so one minus that goes down. Putting both of these parameters on the right-hand-side of the FOC, the right-hand-side goes up. To maintain optimality, then, the firm would want to adjust output to increase its marginal product of the input. For a firm with DRS technology, however, this requires reducing output. Hence, this firm would not wish to respond to the demand shock. The opposite, however, is true of a firm operating an increasing returns technology, which is why we need to assume there are some IRTS firms in each sector. Finally, this result generalizes easily to the case of multiple inputs.

In a mixed RTS economy, with some firms operating IRTS technologies and some DRS or CRS technologies, the IRTS firms will respond to the shock by increasing output. In this case, though, something interesting happens. Consider a DRS firm. This firm will find its share of output falling as the IRTS firms increasing production. It will also see its share of the wage bill falling. In this case, the right-hand-side of the FOC above will be smaller, so the firm’s optimal response will be to make its marginal product smaller, which it does by increasing output. Hence, we have a scenario in which it is possible to think of a leader-follower equilibration, which has IRTS firms moving first in response to changes in aggregate demand, followed by adjustments among the non-IRTS firms. This result also suggests that empirically, we should see market shares increase at the start of a recovery from a downturn, followed by reversions of market shares to "average" levels as the recovery proceeds. We will show below that there is a close relationship between firms’ market shares in this model and the markup of prices over marginal costs, which are considerably easier to observe. Finally, we note that at least in low dimensional or partial equilibrium settings, we can characterize the Nash equilibrium via standard reaction function analysis.
For an example of the reaction function analysis, we consider a single sector with two firms. Firm 1’s output is given by

\[ q_1 = A\phi_1^2 \]

while firm 2’s output is

\[ q_2 = \phi_2^\alpha, \quad 0 < \alpha \leq 1. \]

Total output for the sector is then \( q = q_1 + q_2 \), and total inputs for the sector are \( \phi_1 + \phi_2 = \omega \), for some fixed amount of the input resource \( \omega \). Letting

\[ \gamma_i = \frac{1}{1 - s_i}, \quad i = 1, 2 \]

where \( s_i \) is firm \( i \)'s market share of output, and

\[ \delta_i = \frac{1}{1 - \ell_i}, \quad i = 1, 2 \]

where \( \ell_i \) is firm \( i \)'s share of the inputs, and assuming that the ratio of prices is fixed (for convenience at 1), the first-order conditions for each firm’s best response are

\[ 2A\omega \left[ 1 - \frac{1}{\delta_1} \right] - \gamma_1\delta_1 = 0 \]

for firm 1, and

\[ \alpha \left[ \frac{\gamma_1}{\gamma_1} - \left[ \frac{\delta_1}{\delta_1 - 1} \right] \right]\left[ \frac{\omega}{\delta_1} \right]^{1-\alpha} = 0 \]

for firm 2. The graph below shows the two reaction functions in \( \gamma_1 \) and \( \delta_1 \) generated for the example, with \( A = 30 \), \( \omega = 10 \), and \( \alpha = 0.8 \).
It remains, then, to show that the implicit function theorem (or, more generally, a transversality result) will apply in the neighborhood of the Nash equilibrium for an economy under slack. The Jacobian matrix for the mapping defined by the equilibrium conditions has $M(\ell + 1) + JN + \ell + N$ rows, and $M(\ell + 1) + \sum_{j=1}^{\ell} (K_j - 1) + (J - 1) N + \ell + N$ columns. Calculating the Jacobian yields a matrix of the form

\[
JACOBIAN = \begin{bmatrix}
H_1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & H_M & 0 & 0 & 0 & 0 \\
J & \cdots & J & \bar{Q}_k & \cdots & \bar{Q}_k & 0 & 0 \\
0 & \cdots & 0 & G_{k_1} & \cdots & 0 & \ast & \ast \\
0 & \cdots & 0 & \ast & \cdots & \ast & \ast & \ast \\
0 & \cdots & 0 & 0 & \cdots & 0 & \bar{Q}^{-1} & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \bar{E}^{-1}
\end{bmatrix}
\]

Here,

\[
H_i = \begin{bmatrix}
D^2 u_i & -p \\
-p^T & 0
\end{bmatrix}
\]

for $i = 1, \ldots, M$. 

20
(i.e. the derivatives of the consumer FOCs with respect to \( x_i \) and \( \lambda_i \));

\[
\mathbf{J} = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}
\]

is the matrix of derivatives of \( x_i \) with respect to \( x_i \) and \( \lambda_i \);

\[
\tilde{\mathbf{Q}}_{kj} = \begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & -Q_j & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

is the matrix of derivatives with respect to \( \frac{Q^j_{kj}}{Q^j} \) and \( \frac{W^w_{kj}}{W^w} \) of the vector \( \tilde{\mathbf{Q}}_{kj} \), and

\[
\mathbf{G}_{kj} = \begin{bmatrix} p^j Df_{kj} & -p^j \frac{Q^j_{kj}}{Q^j} \left[ D^2 f_{kj} - \frac{1}{Q^j} Df Df^T \right] \hat{E} + \left[ \hat{W} \hat{W}^{-1}_{kj} \right]^2 \hat{r} \end{bmatrix}
\]

(i.e. the derivatives of firm FOCs with respect to \( \frac{Q^j_{kj}}{Q^j} \) and \( \frac{W^w_{kj}}{W^w} \); here \( \hat{r} = \text{diag } r \) and \( \hat{E} = \text{diag } E \), and \( \ast \) denote derivatives with respect to \( B \) and \( W \) that we don’t need to worry about. Note that

\[
-\frac{Q^j_{kj}}{Q^j} \left[ D^2 f_{kj} - \frac{1}{Q^j} Df Df^T \right] \hat{E} + \left[ \hat{W} \hat{W}^{-1}_{kj} \right]^2 \hat{r}
\]

is positive definite.

Since the Jacobian matrix can be reduced to a square matrix in block upper-triangular form, it is easily verified that the matrix has full row rank, once we note that in the sub-matrices \( \mathbf{G}_{kj} \) the derivatives with respect to \( \frac{Q^j_{kj}}{Q^j} \) ensure that even if some firm’s technology exhibits constant returns to scale (in which case the second derivative matrix would be singular in the relative price direction), there will be a non-zero perturbation possible in the direction of the relative price vector. Hence, we have the following

**Theorem 1** When total input resources are variable, there exist there are solutions to the equilibrium equations in variables consisting of household consumptions and Lagrange multipliers, and firm input wage bill shares, and firm output market shares, in a neighborhood of any given input constrained Nash equilibrium for the economy.

5.3 Discussion

This result clearly implies that firms can maintain Nash equilibrium in the market game via output adjustments alone, without any need for changes in relative prices. So, any reason firms might have to maintain prices will be supported in the model.

The need for adjustments in the units of account normalization to maintain nominal prices suggests a role for monetary policy in providing the degrees of
freedom for making these adjustment. This also suggests that some of the price adjustments delineated in the Klenow-Malin stylized facts reflect adjustments in nominal prices needed to maintain stability of relative prices, but this is obviously open to further study.

Finally, on the issue of the mechanism for price stickiness if we view the data as rejecting menu costs, we would conjecture that learning issues can provide a compelling reason for firms to maintain the kind of "reference" prices observed in the empirical examinations of price dynamics.

The problems with finding effective mechanisms for implementing equilibrium prices in competitive economies are well-known. Scarf’s example is well-known, and shows that the presence of strong income effects can make simple price adjustment dynamics like the Walrasian tatonnement process ineffective. While the market game does provide an explicit price formation mechanism via the ratio of expenditure flows to quantity flows, Kumar and Shubik have shown that the market game is not immune to Scarf-like problems for simple adjustment dynamics akin to tatonnement. There are a series of strong results in the literature on evolutionary game theory showing that when the Nash equilibrium to a game is strict (i.e. equilibrium best-responses are unique), then fitness based dynamics in which better responses to other agents’ play are imitated lead to convergence to the Nash equilibrium. These results haven’t received much attention in conventional general equilibrium analysis or related work in macroeconomics because of the time-scales on which these dynamics operate, and the often non-market-based nature of the interactions generating the convergence.

What the evolutionary game theory results do suggest, particularly in light of the fundamental problems introduced by income effects, is that equilibrium, whether Nash or competitive, is something that must be learned rather than mechanically implemented. The relatively more complex nature of evolutionary learning as opposed to simple mechanical price adjustment processes makes attaining equilibrium costly, as with the menu price friction, but it has the potential to go significantly beyond the menu price model in explaining what we see in the data. If we focus, in particular, on the notion of an evolutionarily stable equilibrium – these occur when the Nash equilibrium is immune to small deviations from best response strategies – we can explain the phenomenon of reference prices observed in the data, and in fact view these as expressing the persistence of relative prices observed in the data (which are, after all, the things that have to be learned in equilibrium). The market game model’s lack of restrictions on equilibrium prices, once obtained, is then consistent with the observation of sales events which can be interpreted as a manifestation of inventory management processes. Finally, the interpretation of the Nash equilibrium as an evolutionarily stable object is also consistent with the empirical data on the large numbers of new businesses that fail within three to five years of launch. In this interpretation, if we view new business ventures (and entrepreneurial activity more generally) as experimentation, the evolutionary stability of equilibrium manifests itself as an immunity to this experimentation.

Finally, as we noted above, there is a close relationship between the market shares of firms in the market game, and the markup over marginal cost in the
model that makes it relatively easy to evaluate the implications of the model for the observable behavior of the markup. The starting point for this analysis is the firm’s cost-minimization problem

\[
\min_w \; \ell \cdot w \\
\text{subject to} \\
\lambda W_1^W \geq q
\]

The first-order conditions for the problem are

\[
\ell^T - \lambda Df^T \hat{W}^{-2} \hat{W}_{-k} \hat{E} = 0.
\]

After some algebra, this can be reduced to

\[
r^T \hat{W} \hat{W}_{-k}^{-1} - \lambda Df^T = 0.
\]

Since the Lagrange multiplier in the cost-minimization problem is just the marginal cost, if we assume that the production function is homogeneous of degree \(\delta\), then by direct calculation we have that

\[
MC(q) = \lambda = \frac{1}{\delta q} r^T \hat{W} \hat{W}_{-k}^{-1} \varphi(q).
\]

To calculate the markup, we note from the first-order conditions for profit maximization that

\[
\hat{W} \hat{W}_{-k}^{-1} v = p \frac{Q_{-k}}{Q} Df
\]

Combining profit maximization and cost minimization results, we have

\[
\lambda = \frac{1}{\delta q} p \frac{Q_{-k}}{Q} Df \cdot \varphi(q) \\
= p \frac{Q_{-k}}{Q}.
\]

Hence

\[
\frac{p}{\lambda} = \frac{Q}{Q_{-k}} \\
= \frac{1}{1 - \frac{Q}{Q} k}.
\]

From this expression, then, it follows that if firm \(k\) increasing output (other firms outputs not changing), then the firm’s market share increases, which leads to an increase in both terms in the markup equation, implying a pro-cyclic change in markups. Unlike the case with standard NK models based on the Dixit-Stiglitz monopolistic competition model, the marginal cost in the market game model is determined via the general equilibrium interactions of technology and demand.
6 References


